# Balanced Tripartite Entanglement, the Alternating Group A4 and the Lie Algebra sl(3,C) ⊕ u(1)

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# Balanced Tripartite Entanglement, the Alternating Group $A_4$ and the Lie Algebra $sl(3,\mathbb{C})\oplus u(1)$

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**Abstract.** We discuss three important classes of three-qubit entangled states and their encoding into quantum gates, finite groups and Lie algebras. States of the GHZ and W-type correspond to pure tripartite and bipartite entanglement, respectively. We introduce another generic class B of three-qubit states, that have balanced entanglement over two and three parties. We show how to realize the largest cristallographic group $W(E_8)$ in terms of three-qubit gates (with real entries) encoding states of type GHZ or W [M. Planat, *Clifford group dipoles and the enactment of Weyl/Coxeter group $W(E_8)$ by entangling gates*, Preprint 0904.3691 (quant-ph)]. Then, we describe a peculiar "condensation" of $W(E_8)$ into the four-letter alternating group $A_4$, obtained from a chain of maximal subgroups. Group $A_4$ is realized from two B-type generators and found to correspond to the Lie algebra $sl(3,\mathbb{C})\oplus u(1)$. Possible applications of our findings to particle physics and the structure of genetic code are also mentioned.

## 1. Introduction

Tripartite aggregates and interactions frequently occur in the natural world. As a first example, it is well known that ordinary matter consists of atoms whose nuclei are made of protons and neutrons, which are themselves made of the lighest quarks $u$ and $d$. A proton consists of a triplet $uud$ and a neutron consists of a triplet $ddu$. Thus, our present universe is made of three types of stable particles, of spin $\frac{1}{2}$, i.e. electrons $e$ and $u$ and $d$ quarks. According to the standard model, there also exists four heavier quarks (among them the strange spin $\frac{1}{2}$ quark $s$), that combine to form unstable composite particles called hadrons, in quark-antiquark pairs (mesons) or three-quark states (baryons). Mathematically, these composite particles are described using the representations of the Lie algebra $su(3)$, in a model named the eightfold way by Gell-Mann and Ne'eman [1]. An old instance goes back to the beginning of chemistry. Among the numerous precursors of Mendeleev, Döbereiner was the first to classify chemical elements into triads [2].

A second relevant example is the genetic code (or amino acid code), that refers to the system of passing from DNA and RNA into the synthesis of proteins. It was discovered in 1961 by Crick *et al.* that the genetic code is a triplet code, made of elementary units of information called codons. There are 64 codons made of four building block bases $A$, $U$, $G$ and $C$ that encode 20 aminoacids. A chain of subalgebras of the Lie algebra $sp(6)$ was proposed for explaining the high degeneracy of the code [3]. See also the modeling of the genetic code based on quantum groups in [4] and related papers.

Our third example is quantum information theory. The term *black hole analogy* has been coined for featuring the relationship between some stringy black hole solutions and three-qubit states [5, 6]. Presumably, this analogy stems from the structure of the largest cristallographic group $W(E_8)$, of cardinality 696 729 600, which one of the authors succeeded in representing in terms of several three-qubit gates [7].

Among the various forms of three-qubit entanglement, a first classification based on SLOCC (stochastic local operations and classical communications) leads to entangled states of the type GHZ and W. The former possess pure (and maximal) three-qubit entanglement and any tracing out about one party destroys all the entanglement. The latter possess equally distributed (and maximal) bipartite entanglement, but no tripartite entanglement. A finer classification is based on local unitary equivalence [8]. In this paper, we are especially interested in a class of entangled three-qubit states displaying equally distributed entanglement about three and two parties. Such states were already encountered in the context of CPT symmetry [9]. Here, they occur when one "condenses" the three-qubit representation of $W(E_8)$ to the alternating group $A_4$, through an appropriate chain of maximal subgroups. The Lie subalgebra of rationals obtained from the generators of $A_4$ is found to be $sl(3,\mathbb{C})\oplus u(1)$. Going upstream in the group sequence, one arrives at a representation of the symmetric group $S_4$, with attached Lie algebra $sl(3,\mathbb{C}) \oplus sl(2,\mathbb{C}) \oplus u(1) \oplus u(1)$, that may play a role in the understanding of elementary particles [10, 11].

In this paper, we expose our new findings about $B$-type entanglement (Sec. 2), the generation of $W(E_8)$ with entangling matrices and the embedding of specific permutation groups $S_4$ and $A_4$ (Sec. 3). A novel three-qubit realization of $sl(3,\mathbb{C})\oplus u(1)$ and its generalization is described in Sec. 4. A rudimentary explanation of Lie groups and algebras is given in the appendix.

Many calculations are performed by using the abstract algebra software Magma [12]. A few papers relating Lie algebras and quantum information theory have already been published [13]-[17].

## 2. $B$-type three-qubit quantum entanglement

One efficient measure of two-qubit entanglement is the tangle $\tau = C^2$, where the concurrence reads $C(\psi) = |\langle\psi|\tilde{\psi}\rangle|$. The flipped transformation $\tilde{\psi} = \sigma_y\,|\psi^*\rangle$ applies to each individual qubit and the spin-flipped density matrix $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$ follows [18]. Explicitly,

$$C(\rho) = \max\left\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\right\},$$

where the $\lambda_i$ are (non-negative) eigenvalues of the product $\rho\tilde{\rho}$, ordered in decreasing order.

Roughly speaking, two pure multiparticle quantum states may be considered as equivalent if both of them can be obtained from the other by means of stochastic local operations and classical communication (the SLOCC group) [19]. There are essentially two inequivalent classes of three-qubit entangled states, with representative $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ (for the GHZ class) and $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$ (for the $W$-class). For *measuring* the entanglement of a triple of quantum systems $A$, $B$ and $C$, one may calculate the amount of *true* three-qubit entanglement from the SLOCC invariant three-tangle [21]

$$\tau^{(3)} = 4\,|d_1 - 2d_2 + 4d_3|\,,$$
$$d_1 = \psi_{000}^2\psi_{111}^2 + \psi_{001}^2\psi_{110}^2 + \psi_{010}^2\psi_{101}^2 + \psi_{100}^2\psi_{011}^2,$$
$$d_2 = \psi_{000}\psi_{111}(\psi_{011}\psi_{100} + \psi_{101}\psi_{010} + \psi_{110}\psi_{001})$$
$$+\psi_{011}\psi_{100}(\psi_{101}\psi_{010} + \psi_{110}\psi_{001}) + \psi_{101}\psi_{010}\psi_{110}\psi_{001},$$
$$d_3 = \psi_{000}\psi_{110}\psi_{101}\psi_{011} + \psi_{111}\psi_{001}\psi_{010}\psi_{100},$$

as well as the amount of two-qubit entanglement between two parties, by tracing out over partial subsystems $AB$, $BC$ and $AC$.

For a two-qubit state $|\psi\,\rangle = \alpha\,|00\,\rangle + \beta\,|01\,\rangle + \gamma\,|10\,\rangle + \delta\,|11\,\rangle$, the concurrence is $C = 2\,|\alpha\delta - \beta\gamma|$, and thus satisfies the relation $0 \leq C \leq 1$, with $C = 0$ for a separable state and $C = 1$ for a maximally entangled state.

The three-qubit entangled state $|GHZ\rangle$ is maximally entangled, with three-tangle $\tau^{(3)} = 1$ and all two-tangles vanishing; that is, whenever one of the qubits is traced out, the remaining two are completely unentangled. On the other hand, the entangled state $|W\rangle$ has $\tau^{(3)} = 0$, but it maximally retains bipartite entanglement [19].

Refinements on the above classification may be obtained if one classifies the three-qubit state up to local unitary equivalence (the LU group) [8]. Thus, if one singles out the first party $A$, a generic state of three qubits depends, up to LU, on five parameters

$$|\psi\rangle = \lambda_0\,|000\rangle + \lambda_1 e^{i\phi}\,|100\rangle + \lambda_2\,|101\rangle + \lambda_3\,|110\rangle + \lambda_4\,|111\rangle\,,$$

$$\lambda_i > 0,\ \sum_{j=0}^{4}\lambda_i^2 = 1 \text{ and } 0 \leq \phi \leq \pi.$$

In the sequel, we are interested in entangled states of the $B$-class, where $\lambda_1 = 0$, with a representative

$$|B\rangle = \tfrac{1}{2}(|000\rangle + |101\rangle + |110\rangle + |111\rangle). \tag{1}$$

The three-tangle of the $B$-state is $\tau^{(3)} = \frac{1}{4}$ and the density matrices of the bipartite subsystems are

$$\rho_{BC} = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix},\ \rho_{AB} = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix},\ \rho_{AC} = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix}.$$

The set of eigenvalues $\left\{\frac{1}{16}(3+2\sqrt{2}), \frac{1}{16}(3-2\sqrt{2}), 0, 0\right\}$ is uniform over the subsystems with two-tangles $\tau_{AB} = \tau_{AC} = \tau_{BC} = \frac{1}{4}$. Similarly, the linear entropies $\tau_{A(BC)} = \tau_{B(AC)} = \tau_{C(AB)} = \frac{3}{4}$ are the same (see [18] for the meaning of linear entropies such as $\tau_{A(BC)} = \tau^{(3)} + \tau_{AB} + \tau_{AC}$). Thus, the entanglement measure for two parties equals the entanglement measure for three parties. This equal balance of the entanglement for two or three parties justifies our notation for the $B$-class ‡.

## 3. Three-qubit entanglement and the crystallogaphic group $W(E_8)$

Recently, by studying the Clifford group on two and three qubits, we discovered several eight-dimensional orthogonal realizations of the largest crystallographic group $W(E_8)$, and of its relevant subgroups. As described in papers [7, 9], these representations find their kernel in two-qubit entanglement and the following orthogonal matrix

$$S_2 = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} + & - & - \\ - & + & - \\ - & - & + \\ + & + & + \end{pmatrix}, \tag{2}$$

that encodes the joint eigenstates of the triple of observables.

$$\left\{\sigma_x \otimes \sigma_z, \sigma_z \otimes \sigma_x, \sigma_y \otimes \sigma_y\right\}. \tag{3}$$

Rows of the second matrix contain the sign of eigenvalues $\pm 1$ of the triple of observables, and a row of the first matrix corresponds to a joint eigenstate [e. g. the first row corresponds to the state $\frac{1}{2}(|00\rangle - |01\rangle + |10\rangle + |11\rangle$ with eigenvalues $(1, -1-1)$].

To abound in this claim, let us consider the following triple of three-qubit observables

$$\sigma_z \otimes \left\{\sigma_x \otimes \sigma_z, \sigma_z \otimes \sigma_x, \sigma_y \otimes \sigma_y\right\}, \tag{4}$$

‡ The $B$-states are denoted CPT states in our previous work [9]. Choudhary and coworkers [20] computed the local realistic violation of the inequality (given in Eq. (3) of their paper) for the generic state $|B\rangle$ and found the value 0.608723, a big violation compared to 0.175459 for the generic GHZ state and 0.192608 for the generic $W$ state.

that follows from (3) by adjoining the tensor product $\sigma_z$ at the left hand side. Eigenstates of (4) may be used for encoding the rows of the following orthogonal matrix

$$S_3 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & 1 \end{pmatrix}, \tag{5}$$

and to generate the derived subgroup $W'(E_8) \cong O^+(8,2)$ of order 348 364 800 [recall that $O^+(8,2)$ is the general eight-dimensional orthogonal group over $GF(2)$]

$$W'(E_8) \cong \langle \sigma_x \otimes S_2, S_3 \rangle \,. \tag{6}$$

Replacing the $S_3$ state by the GHZ-type generator $b$ whose explicit form is given by Eq. (18) of [7], one gets $W'(E_7) \cong \langle \sigma_x \otimes S_2, b \rangle$. Indeed many important subgroups of $W(E_8)$ may be realized by means of the appropriate orthogonal generators.

Here one focuses on a sequence of subgroups leading to a specific representation of the four-letter alternating group $A_4$ (as well as the symmetric group $S_4$) and a representation of the Lie algebra $sl(3,\mathbb{C})$ (as well as its more general parent). The relevant sequence is

$$W'(E_8) \supset W'(E_7) \supset W'(E_6) \supset G_{648} \supset S_4 \supset A_4. \tag{7}$$

Starting from $W'(E_8)$ [as in (6)], one looks at the maximal subgroups. One of the three subgroups of the largest cardinality is isomorphic to $W(E_7)$, of order 2 903 040 §. Then, in the derived subgroup $W'(E_7)$, one takes the largest maximal subgroup $W(E_6)$, of order 51 840. Among the five maximal subgroups of $W'(E_6)$, two of them have the cardinality 648; one selects the one isomorphic to the semi-direct product $G_{648} = Z_2^7 \rtimes S_4$ ‖. Finally, one is interested in the subgroup $S_4$ of $G_{648}$, as well as in its derived subgroup $A_4$.

The alternating group $A_4$ may be realized by means two orthogonal generators $x_{A_4}$ and $y_{A_4}$, whose rows are similar up to a permutation, and encode three-qubit states of the $B$-type, with similar two- and three-tangles as it results from straightforward calculations.

§ The second largest subgroup of $W(E_8)$ is the real Clifford group $\mathcal{C}_3^+$, of order 2 580 480 studied in [7, 9].

‖ The maximal subgroup of the largest cardinality in $W'(E_6)$ is isomorphic to the perfect group $M_{20} = Z_2^4 \rtimes A_5$ of order 960, and is described in [23, 24].

$$x_{A_4} = \frac{1}{2}\begin{pmatrix} 0 & 1 & -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ -1 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 & -1 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 & -1 & -1 & 0 & -1 \\ 0 & 1 & -1 & 1 & 0 & 0 & -1 & 0 \end{pmatrix},$$

$$y_{A_4} = \frac{1}{2}\begin{pmatrix} 0 & -1 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ -1 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 & -1 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 & -1 & -1 & 0 & -1 \\ 0 & -1 & 1 & 1 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{8}$$

The relationship between the finite group $A_4$ and the Lie algebra $sl(3, \mathbb{C})$ is established in Sec. 4.

## 4. The Lie algebra of $sl(3, \mathbb{C})$: old and new

Group operations we considered in our earlier papers were finite group operations. We are now interested in group operations which are smooth, yet still compatible with the finite symmetries. This is where the concept of a Lie group, endowed with its Lie algebra of commutation relations, enters the game. For quantum mechanics, the favourite Lie group is the matrix Lie group $SL(n, \mathbb{C})$. For an introduction to Lie groups and Lie algebras see [25]–[28], and the appendix of this paper.

*Standard representation of* $sl(3, \mathbb{C})$

It is well know that $sl(3, \mathbb{C})$ occurs in the context of particle physics for representing quark states. It is part of the standard model of elementary particles $su(3) \oplus su(2) \oplus u(1)$ [1]. Remarkably, one arrives at a form reminiscent of the standard model in representing the Lie algebra attached to groups $A_4$ and $S_4$, as given in Sec. (3).

A Chevalley basis for the algebra $sl(3, \mathbb{C})$ may be written as

$$x_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ x_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ x_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$y_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ y_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ y_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$h_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},\ h_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and the corresponding table of commutators reads

$$\left(\begin{array}{c|ccc|ccc|cc}
[.,.] & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & h_1 & h_2 \\
\hline
x_1 & . & -x_3 & . & h_1 & . & y_2 & -2x_1 & x_1 \\
x_2 & & . & . & . & h_2 & -y_1 & x_2 & -2x_2 \\
x_3 & & & . & x_2 & -x_1 & h_1+h_2 & -x_3 & -x_3 \\
\hline
y_1 & & & & . & y_3 & . & 2y_1 & -y_1 \\
y_2 & & & & & . & . & -y_2 & 2y_2 \\
y_2 & & & & & & . & y_3 & y_3 \\
\hline
h_1 & & & & & & & . & . \\
h_2 & & & & & & & & .
\end{array}\right). \tag{9}$$

Using this table, the positive roots relative to the pair of generators $H = (h_1, h_2)$ are easily discerned as $\alpha_1 = (2, -1)$, $\alpha_2 = (-1, 2)$ and $\alpha_3 = (1, 1)$, corresponding to the root vectors $x_1$, $x_2$ and $x_3$, respectively (see [25] for details ¶). Negative roots have opposite signs. The Killing matrix is of the form

$$\mathrm{Kil} = 6\begin{pmatrix}
2 & . & . & . & 1 & . & . & . \\
. & . & . & 1 & . & . & . & . \\
. & . & . & . & . & . & 1 & . \\
. & 1 & . & . & . & . & . & . \\
1 & . & . & . & 2 & . & . & . \\
. & . & . & . & . & . & . & 1 \\
. & . & 1 & . & . & . & . & . \\
. & . & . & . & . & 1 & . & .
\end{pmatrix}. \tag{10}$$

The adjoint representation provides another representation of the Lie algebra $sl(3, \mathbb{C})$

$$ad_{x_1} = \begin{pmatrix}
. & . & . & . & . & . & -2 & 1 \\
. & . & . & . & . & . & . & . \\
. & -1 & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . \\
. & . & . & . & . & 1 & . & . \\
. & . & . & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . \\
. & . & . & . & . & . & . & .
\end{pmatrix}, \quad ad_{x_2} = \begin{pmatrix}
. & . & . & . & . & . & . & . \\
. & . & . & . & . & . & 1 & -2 \\
1 & . & . & . & . & . & . & . \\
. & . & . & . & . & -1 & . & . \\
. & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . \\
. & . & . & . & 1 & . & . & .
\end{pmatrix},$$

$$ad_{x_3} = \begin{pmatrix}
. & . & . & . & -1 & . & . & . \\
. & . & . & . & 2 & . & . & . \\
. & . & . & . & . & . & -1 & -1 \\
. & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . \\
. & . & . & . & . & 1 & . & . \\
. & . & . & . & . & 1 & . & .
\end{pmatrix}, ad_{y_1} = \begin{pmatrix}
. & . & . & . & . & . & . & . \\
. & . & -1 & . & . & . & . & . \\
. & . & . & . & . & . & . & . \\
. & . & . & . & . & . & 2 & -1 \\
. & . & . & . & . & . & . & . \\
. & . & . & . & 1 & . & . & . \\
-1 & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & .
\end{pmatrix},$$

¶ For instance, since $[h_1, x_1] = 2x_1$ and $[h_2, x_1] = -x_1$, one gets the first root $\alpha_1 = (2, -1)$ corresponding to the root vector $x_1$.

$$ad_{y_2} = \begin{pmatrix} . & . & 1 & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & -1 & 2 \\ . & . & . & -1 & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & -1 & . & . & . & . & . & . \end{pmatrix}, ad_{y_3} = \begin{pmatrix} . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . & . \\ -1 & . & . & . & . & . & . & . \\ . & . & . & . & . & . & 1 & 1 \\ . & . & -1 & . & . & . & . & . \\ . & -1 & . & . & . & . & . & . \end{pmatrix},$$

$$ad_{h_1} = \begin{pmatrix} 2 & . & . & . & . & . & . & . \\ . & -1 & . & . & . & . & . & . \\ . & . & 1 & . & . & . & . & . \\ . & . & . & -2 & . & . & . & . \\ . & . & . & . & 1 & . & . & . \\ . & . & . & . & . & -1 & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{pmatrix}, \quad ad_{h_2} = \begin{pmatrix} -1 & . & . & . & . & . & . & . \\ . & 2 & . & . & . & . & . & . \\ . & . & 1 & . & . & . & . & . \\ . & . & . & 1 & . & . & . & . \\ . & . & . & . & -2 & . & . & . \\ . & . & . & . & . & -1 & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{pmatrix}.$$

Using the Cartan subalgebra $(h'_1, h'_2)$, with $h'_1 = \text{diag}(1, ., 1, -1, ., -1, ., .)$ and $h'_2 = \text{diag}(., 1, 1, ., -1., -1, ., .)$, the new system of positive roots is computed as $\{(1,0), (0,1), (1,1)\}$.

*Representation of $sl(3, \mathbb{C})$ stemming from $A_4$*

Let us now go back to tripartite quantum entanglement and show how the $B$-states (1) are related to a new representation of $sl(3, \mathbb{C})$.

Using Magma, we created a (real) subalgebra of the matrix Lie algebra defined over the rational field, that is obtained from the generators of the finite group $A_4$ described in (8). The algebra is found to be isomorphic to the Lie algebra $\mathfrak{g}_{A_4}$ of type $sl(3, \mathbb{C}) \oplus u(1)$, and the derived algebra $\mathfrak{g}'_{A_4} = [\mathfrak{g}_{A_4}, \mathfrak{g}_{A_4}]$ turns out to be isomorphic to $sl(3, \mathbb{C})$.

A Chevalley basis of the algebra $\mathfrak{g}'_{A_4}$ is as follows

$$x_1 = \begin{pmatrix} . & . & . & . & . & . & . & . \\ . & . & . & 1 & . & . & 1 & . \\ . & . & . & -1 & . & . & -1 & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{pmatrix}, \ x_2 = \begin{pmatrix} . & 1 & -1 & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & 1 & -1 & . & . & . & . & . \end{pmatrix}, \ x_3 = 2 \begin{pmatrix} . & . & . & 1 & . & . & 1 & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & 1 & . & . & 1 & . \end{pmatrix},$$

$$y_1 = \frac{1}{4} \begin{pmatrix} . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & 1 & -1 & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & 1 & -1 & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{pmatrix}, \ y_2 = \frac{1}{4} \begin{pmatrix} . & . & . & . & . & . & . & . \\ 1 & . & . & . & . & . & . & 1 \\ -1 & . & . & . & . & . & . & -1 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{pmatrix}, \ y_3 = \frac{1}{8} \begin{pmatrix} . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 1 & . & . & . & . & . & . & 1 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 1 & . & . & . & . & . & . & 1 \\ . & . & . & . & . & . & . & . \end{pmatrix},$$

$$h_1 = \frac{1}{2} \begin{pmatrix} . & . & . & . & . & . & . & . \\ . & 1 & -1 & . & . & . & . & . \\ . & -1 & 1 & . & . & . & . & . \\ . & . & . & -1 & . & . & -1 & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & -1 & . & . & -1 & . \\ . & . & . & . & . & . & . & . \end{pmatrix}, \quad h_2 = \frac{1}{2} \begin{pmatrix} 1 & . & . & . & . & . & . & 1 \\ . & -1 & 1 & . & . & . & . & . \\ . & 1 & -1 & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 1 & . & . & . & . & . & . & 1 \end{pmatrix}.$$

and its elements are readily seen to fit into the table of commutators (9) of $sl(3,\mathbb{C})$. As a result, the roots relative to a new pair of generators $(h_1, h_2)$ given above are the $\alpha_i$ given in the preceding subsection. This may be a useful feature of the new representation, in contrast to the adjoint one, for subsequent applications to the physics of elementary particles.

Going upstream in the group sequence (7) one arrives at a three-qubit realization of the symmetric group $S_4$. The group $A_4$, with generators as in (8), is the derived subgroup of $S_4$. The corresponding Lie algebra, of dimension 13, may be decomposed as a direct sum of simple Lie algebra as follows

$$\mathfrak{g}_{S_4} = sl(3,\mathbb{C}) \oplus sl(2,\mathbb{C}) \oplus u(1) \oplus u(1), \tag{11}$$

in which the algebra $sl(3,\mathbb{C}) \oplus u(1)$ is embedded.

A basis for the representation of $sl(2,\mathbb{C})$ in (11) is as follows

$$\begin{pmatrix} 1 & . & -1 & . & 1 & -1 & . & . \\ . & -1 & . & . & -1 & 1 & . & 1 \\ -1 & . & 1 & . & -1 & 1 & . & . \\ . & . & . & . & . & . & . & . \\ 1 & -1 & -1 & . & . & . & . & 1 \\ -1 & 1 & 1 & . & . & . & . & -1 \\ . & . & . & . & . & . & . & . \\ . & 1 & . & . & 1 & -1 & . & -1 \end{pmatrix},$$

$$\begin{pmatrix} . & 1 & . & . & 1 & -1 & . & -1 \\ . & -\frac{1}{2} & . & . & -\frac{1}{2} & \frac{1}{2} & . & \frac{1}{2} \\ . & -1 & . & . & -1 & 1 & . & 1 \\ . & . & . & . & . & . & . & . \\ . & \frac{1}{2} & . & . & \frac{1}{2} & -\frac{1}{2} & . & -\frac{1}{2} \\ . & -\frac{1}{2} & . & . & -\frac{1}{2} & \frac{1}{2} & . & \frac{1}{2} \\ . & . & . & . & . & . & . & . \\ . & \frac{1}{2} & . & . & \frac{1}{2} & -\frac{1}{2} & . & -\frac{1}{2} \end{pmatrix}, \quad \begin{pmatrix} . & . & . & . & . & . & . & . \\ 1 & -\frac{1}{2} & -1 & . & \frac{1}{2} & -\frac{1}{2} & . & \frac{1}{2} \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 1 & -\frac{1}{2} & -1 & . & \frac{1}{2} & -\frac{1}{2} & . & \frac{1}{2} \\ -1 & \frac{1}{2} & 1 & . & -\frac{1}{2} & \frac{1}{2} & . & -\frac{1}{2} \\ . & . & . & . & . & . & . & . \\ -1 & \frac{1}{2} & 1 & . & -\frac{1}{2} & \frac{1}{2} & . & -\frac{1}{2} \end{pmatrix},$$

The Killing matrix of the representation may be diagonalized

$$24 \begin{pmatrix} 4 & 1 & 1 \\ 1 & . & 2 \\ 1 & 2 & . \end{pmatrix} := TDT^{-1} \text{ with } D = 96 \begin{pmatrix} 1 & . & . \\ . & -1 & 0 \\ . & . & 3 \end{pmatrix} \text{ and } T := \begin{pmatrix} 1 & . & . \\ -1 & 4 & . \\ 2 & -7 & -1 \end{pmatrix}.$$

corresponding to the representation $su(1,1)$ of $sl(2,\mathbb{C})$, with signature $(2,1)$.

## 5. Conclusion

We have found a new intricate relation between finite group theory, Lie algebras and three-qubit quantum entanglement. In particular, the connection between *balanced* tripartite entanglement (in Sec. 2) and the eight-dimensional representation of the Lie algebra $sl(3,\mathbb{C})$ (in Sec. 5) is put forward. Earlier papers of one of the authors focused on the three-qubit realization of Coxeter groups, such as the largest one $W(E_8)$, together with its most relevant subgroups comprising the three-qubit Clifford group $\mathcal{C}_3^+$, $W(E_7)$, $W(E_6)$, $W(F_4)$ and other subgroups [7, 9]. Here, one discovers that the two-qubit real entangling gate $S_2$ [see Eq. (2)] and its three-qubit parent, the gate $S_3$ [see eq. (3)] are building stones of the realization of $W'(E_8)$. An appropriate reduction of $W'(E_8)$ to the four-letter alternating group $A_4$ [see (7)] is used to represent the algebra $\mathfrak{g}_{A_4} = sl(3,\mathbb{C}) \oplus u(1)$. The parent of $A_4$ is the symmetric group $S_4$ and the corresponding Lie algebra is $\mathfrak{g}_{S_4} = sl(3,\mathbb{C}) \oplus sl(2,\mathbb{C}) \oplus u(1) \oplus u(1)$, which reminds us of the standard model of particles [10]. From a mathematical point of view, the relation to algebraic surfaces is worthwhile to be investigated in the future [30]. As an interesting implication for biosciences, the four letters occurring in the permutation groups $A_4$ and $S_4$ suggest to consider $\mathfrak{g}_{S_4}$ algebra as a new candidate for a deeper insight into the degeneracies of genetic code.

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## Appendix: Elements on Lie groups and Lie algebras

Let $G$ be a matrix Lie group, the Lie algebra $\mathfrak{g}$ of $G$ is real and defined as the set of all matrices $X$ such that $e^{tX}$ is in $G$ for all real numbers $t$. There is an important property that

$$\text{for any } X \in \mathfrak{g}, \text{ and for } A \in G,\ \mathrm{Ad}_A(X) = AXA^{-1} \in \mathfrak{g},$$

i. e. conjugation of an element of the Lie algebra by an element of the Lie group preserves the algebra. The above map from the Lie algebra to itself is called the adjoint mapping.

This definition is reminiscent of the definition of the Clifford group $\mathcal{C}$, that is defined as the normalizer of the Pauli group $\mathcal{P}$ within the unitary group $U(n)$, i.e. denoting $X$ an arbitrary error arising from the Pauli group, and $A$ an element of the Clifford group [7, 23, 24]

$$\text{then for any } X \in \mathcal{P}, \text{ and for } A \in \mathcal{C} \subset U(n),\ AXA^{-1} \in \mathcal{P}.$$

In some sense, Lie groups and algebras are a smooth (continuous) formulation of quantum error correction.

The Lie algebra is endowed with a map (called commutator) $[.,.]$ from $\mathfrak{g} \times \mathfrak{g}$ to $\mathfrak{g}$, with the properties

$$(i)\ [.,.] \text{ is bilinear}$$

$(ii)$ $[X, Y] = -[Y, X]$ for all $X, Y \in \mathfrak{g}$ (anticommutativity)

$(iii)$ $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for all $X, Y, Z \in \mathfrak{g}$ (Jacobi identity).

The adjoint endomorphism "Ad" can be reformulated in terms of commutators by the linear map "ad" as follows

$$\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g} \text{ defined by } \mathrm{ad}_X(Y) = [X, Y].$$

Thus, the map "ad" from $X$ to $\mathrm{ad}_X$ is a linear map from $\mathfrak{g}$ to the space $gl(\mathfrak{g})$ of linear operators from $\mathfrak{g}$ to $\mathfrak{g}$, and there exists a Lie algebra homomorphism $\mathfrak{g}$ to $gl(\mathfrak{g})$ by the relation

$$\mathrm{ad}_{[X,Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y].$$

Selecting a basis $X_1, \ldots, X_n$ of the $n$-dimensional Lie algebra, for each $i$ and $j$ one obtains

$$[X_i, X_j] = c_{ij}^k X_k,$$

in which the *structure constants* $c_{ij}^k$ (with respect to the basis) define the bracket operation on $\mathfrak{g}$.

For a simple real or complex Lie algebra (see below) there exists a basis, called the Chevalley basis, for which the structure constants are relative integers.

A complex Lie algebra $\mathfrak{g}$ is called indecomposable if the only ideals in $\mathfrak{g}$ are $\mathfrak{g}$ and $\{0\}$, it is called simple if it is indecomposable and $\dim(\mathfrak{g}) \geq 2$. The algebra $\mathfrak{g}$ is called reductive if it is isomorphic to a direct sum of indecomposable Lie algebras; it is called semi-simple if it is isomorphic to a direct sum of simple Lie algebras.

A subalgebra of a Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{h}$ of $\mathfrak{g}$ such that $[H_1, H_2] \in \mathfrak{h}$ for all $H_1$ and $H_2 \in \mathfrak{h}$.

If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras, then a linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ is called a *Lie algebra homorphism* if $\phi([X, Y]) = [\phi(x), \phi(y)]$, for all $X, Y \in \mathfrak{g}$. If, in addition, $\phi$ is one-to-one and onto, then $\phi$ is called a *Lie algebra isomorphism.*

The adjoint map "ad" may be used to figure out the geometry of the Lie algebra. The composition of two "ad" defines a symmetric bilinear form called the the Killing form

$$B(X, Z) = \mathrm{trace}(\mathrm{ad}(X)\mathrm{ad}(Z)),$$

that possesses several important properties. It is associative, i. e. $B([X, Y], Z) = B(X, [Y, Z])$; it is invariant under the automorphisms $s$ of the algebra $\mathfrak{g}$ that is, $B(s(X), s(Z)) = B(X, Z)$ for $s$ in $\mathrm{Aut}(\mathfrak{g})$, and the Cartan criterion states that a Lie algebra over a field of characteristic zero is semi-simple iff the Killing form is nondegenerate.

The matrix elements $B_{ij}$ of the Killing form are related to structure constants as follows

$$B_{ij} = \frac{1}{I_{\mathrm{ad}}} c_{im}^n c_{jn}^m,$$

where the *Dynkin index* $I_{\text{ad}}$ depends on the representation.

*Real forms*

A real form is a real Lie algebra $\mathfrak{g}_0$ whose complexification is a complex Lie algebra $\mathfrak{g}$ [26].

Let us define the *signature* of a real Lie algebra as a pair $(a_1, a_2)$, that counts the number of positive $(a_1)$ and negative $(a_2)$ entries in the diagonal form of $B$. In particular, a real Lie algebra $\mathfrak{g}$ is called compact if its Killing form is negative definite. It is also known that a compact Lie algebra corresponds to a compact Lie group.

As an illustrative example, the special linear algebra $sl(2,\mathbb{C})$ has two real forms, the so-called (non-compact) split real form $sl(2,\mathbb{R}) \cong su(1,1)$ of signature $(2,1)$ and the compact real form $su(2)$ of signature $(0,3)$ $^+$. The first real form $sl(2,\mathbb{R})$ follows from the representation of $sl(2,\mathbb{C})$ in the Pauli spin basis, with Killing matrix $4\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and eigenvalues 4, 8 and $-4$. The second real form follows from the adjoint representation [associated to the orthogonal Lie group $SO(3)$]

$$\text{ad}_{\sigma_z} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{ad}_{\sigma_x} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \text{ad}_{\sigma_y} = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix},$$

with Killing matrix $2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

More generally, the algebra $sl(3,\mathbb{C})$ has three real forms, the split real form $sl(3,\mathbb{R})$, the compact real form $su(3)$, and the non-split real form $su(2,1)$.

*Roots*

Let $\mathfrak{g}$ be a complex semi-simple Lie algebra, then a Cartan subalgebra of $\mathfrak{g}$ is a complex subspace $\mathfrak{h}$ of $\mathfrak{g}$ with the following properties

$$(i) \text{ For all } H_1 \text{ and } H_2 \in \mathfrak{h}, [H_1, H_2] = 0,$$

$$(ii) \text{ For all } X \in \mathfrak{g}, \text{ if } [H, X] = 0 \text{ for all } H \in \mathfrak{h}, \text{ then } X \in \mathfrak{h},$$

$$(iii) \text{ For all } H \in \mathfrak{h}, \text{ ad}_H \text{ is diagonalizable}$$

A root of $\mathfrak{g}$, relative to a Cartan subalgebra of $\mathfrak{h}$, is a nonzero linear functional $\alpha$ on $\mathfrak{h}$ such that there exists a nonzero element $X$ of $\mathfrak{g}$ with

$$[H, X] = \alpha(H)X,$$

for all $H$ in $\mathfrak{h}$.

$^+$ The non-compact split real form $e_7(7)$ of the Lie algebra $e_7$ plays a role in [6].

If $\alpha$ is a root, then the *root space* $\mathfrak{g}_\alpha$ is the space of all $X$ in $\mathfrak{g}$ for which $[H, X] = \alpha(H)X$ for all $H$ in $\mathfrak{h}$. The elements of $\mathfrak{g}_\alpha$ are the root vectors for the root $\alpha$. It is convenient to single out the set $\{\alpha_1, \ldots, \alpha_k\}$ of roots, that have the property that all the roots can be expressed as linear combinations of the $\alpha_i (i = 1..k)$. Such roots are called *positive simple roots*.

As it is well known, the geometry of a semi-simple Lie algebra $\mathfrak{g}$ may be made explicit by introducing the *Cartan subalgebra* and its attached root space. The connection of Lie algebras to finite symmetries occurs by looking at the isometry group of the root system. Specifically, the subgroup generated by reflections through the hyperplanes orthogonal to the roots is called the Weyl group $W(\mathfrak{g})$ [also denoted $W(G)$] of the Lie algebra $\mathfrak{g}$ (and of the corresponding Lie group $G$).

Weyl groups are found in the context of quantum error correction [29]. Weyl group $W(E_8)$ and related Weyl subgroups were already encountered in Sec. (3), in a three-qubit realization.

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